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On the high Reynolds number flow over a wavy boundary

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The nature of a shear flow over a wavy boundary of small amplitude is investigated. It is found that if the viscosity is small, the nature of the flow is highly dependent on the wave amplitude. If the wave amplitude is truly infinitesimal, the flow is described by the Orr–Sommerfeld equation and in the neighbourhood of the critical layer viscous stresses are important even in the limit of vanishing viscosity. However, if the wave is sufficiently large, viscous stresses may be neglected even in the critical layer. An approximate solution of the inviscid equations of motion is obtained to describe the flow over a small but finite wave in the limit of infinite Reynolds number.

1. Introduction

The purpose of this note is to examine the two-dimensional laminar shear flow of a nearly inviscid fluid over a boundary that has the form of a small amplitude periodic travelling wave. In his paper concerning the generation of surface waves, Miles (1957) pointed out the dominant role played by the dynamics of the critical layer in which the shear flow velocity is nearly equal to the wave propagation velocity. In his development Miles proposed to determine the nature of the flow in the limits of zero wave amplitude and infinite Reynolds number, where the limits are taken in that order. As a consequence of this limiting procedure, the problem was first reduced to solving the Orr–Sommerfeld equation and then, upon neglecting viscosity, to solving the Rayleigh equation. But as is well known from hydrodynamic stability theory, except in certain special cases, this equation has solutions which are singular at the critical layer height and, in fact, the appropriate condition for matching the solutions on each side of the critical layer cannot be found from Rayleigh's equation alone. Making use of well-known asymptotic solutions of the full Orr–Sommerfeld equation, Miles identified the correct solution to the inviscid equation and thereby determined such important quantities as the pressure work done on the wavy boundary.

Since the results of the Miles theory are sensitively dependent on the manner in which the singular solutions of the Rayleigh equation are matched across the critical layer, it is of at least academic interest to examine this point in detail.

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There is no doubt that the solution given by Miles is the correct one in the joint limits of infinitesimal wave amplitude and vanishing viscosity taken in that order. However, it is reasonable to question the importance of the order in which these limits are taken and it is to this point that the present work is addressed.† In what follows it will be shown that the singularity found in the solution of Rayleigh's equation is not a consequence solely of neglecting the viscous terms in the full equation of motion, but rather, of neglecting both non-linear and viscous terms. Consequently, by retaining certain non-linear terms in the neighbourhood of the critical layer it is possible to construct an approximate solution of the inviscid equations in which the velocity is everywhere continuous and bounded. This solution would presumably apply to the shear flow over a wave of sufficient amplitude that inertial and pressure forces dominate over viscous stresses everywhere including the critical layer.

For the purposes of analysis it is convenient to adopt a dimensionless co-ordinate system which translates at the wave propagation velocity with the y-axis directed upward and the x-axis horizontally. In this reference frame the motion is steady and the dimensionless horizontal velocity, u, is positive above the critical layer and negative below, while the wavy surface is determined by $y = y_s = \epsilon\zeta(x)$, where ϵ is considered to be a small parameter.

For flows in which the viscous stresses are everywhere negligible the equations of motion can be reduced to the statement that the vorticity, ω, is constant along every streamline. Furthermore, since the flow is incompressible, the velocity field may be described by a stream function, ψ. Unfortunately, it is not possible to state the relationship between ω and ψ in functional form because there is not a one-to-one correspondence between streamlines and the value of the stream function. This is so because the stream function has a minimum value, say $\psi = 0$, in the critical layer and $\psi > 0$ may correspond to two different streamlines, one on either side of the critical layer, along which the vorticity has different values. This difficulty may be overcome by introducing the co-ordinates (ξ, η) where $\xi = x$ and η, which is constant along a streamline, is defined implicitly by

$$\frac{d\psi}{d\eta} = \int_0^\eta \omega(\eta')\,d\eta' \equiv W(\eta).$$

This definition ensures that there is a unique correspondence between streamlines and values of η. The co-ordinate η is a measure of the mean elevation of the streamline; if the flow is parallel $\eta = y$. The characteristic parameters used in non-dimensionalization are chosen so that $W'(0) = -W''(0) = 1$; the form of W determines the shape of the mean velocity profile.

The equation of motion may be written as

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \nabla^2\psi = \frac{dW}{d\eta}. \tag{1}$$

† I have recently learned that Prof. D. J. Benney and Dr Bergeron have recently completed an independent investigation of this point and that their results are to be published in *Studies in Applied Mathematics*.

In terms of the co-ordinates (ξ, η) the elevation of a streamline is h, where

$$y = \eta + h(\xi, \eta),$$

the velocity components are

$$u = \frac{W}{1+h_\eta}, \quad v = \frac{Wh_\xi}{1+h_\eta}$$

and the equation of motion is

$$(1+h_\xi^2)\,h_{\eta\eta} + (1+h_\eta)^2 h_{\xi\xi} - 2(1+h_\eta)\,h_\xi h_{\xi\eta} + (W'/W)\{(1+h_\eta)^3 - (1+h_\eta) - h_\xi^2(1+h_\eta)\} = 0. \quad (2)$$

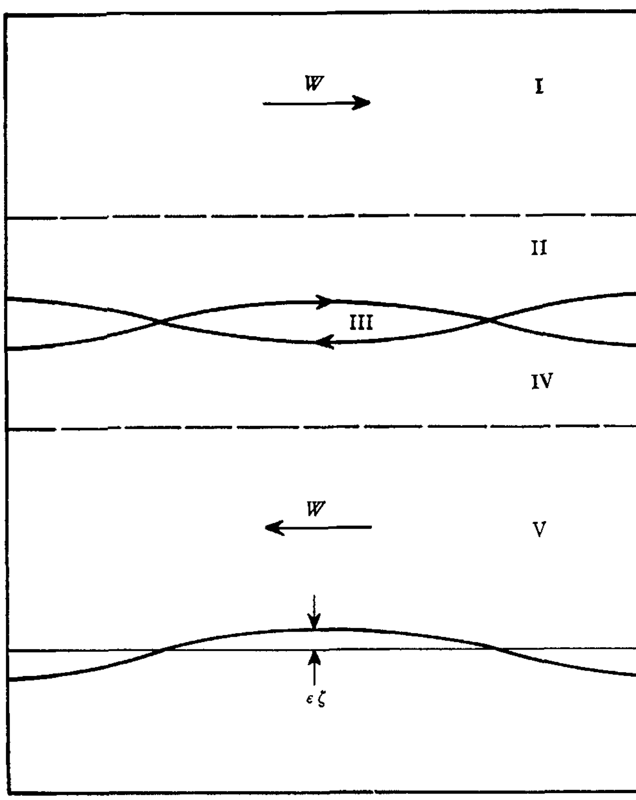

FIGURE 1. The five regions used in the analysis.

The boundary conditions on h are

$$h \to 0 \quad \text{as} \quad \eta \to \infty \quad \text{and} \quad h(\xi, \eta_s) = \epsilon\zeta(\xi). \quad (3)$$

Anticipating the method of analysis, the flow is divided into the five regions which are sketched in figure 1. Regions I and V, which will be referred to as 'outer' regions, are those parts of the flow in which Rayleigh's equation pertains. Regions II and IV are 'inner' regions in which non-linear effects are important even as ϵ tends to zero. Region III is that portion of the flow in which the streamlines are closed. Regions I and II and regions IV and V are, of course, overlapping and will be matched together following the familiar technique of inner and outer expansions. Then the composite regions making up the open streamline flow above and below the critical layer will be joined to the closed streamline region, thereby completing the analysis.

2. Open streamline regions

In this section the open streamline flow in regions I, II, IV and V is examined. While the solutions on each side of the closed streamline region must be considered separately, they will be obtained following nearly parallel developments. In the following section these solutions will be joined to the flow in region III.

In the outer regions I and V the streamline elevation is expanded

$$h = \epsilon H^{(1)} + O(\epsilon^2)$$

and when this is substituted into (2) and terms of $O(\epsilon)$ are collected it is seen that

$$H^{(1)}_{\eta\eta} + H^{(1)}_{\xi\xi} + 2(W'/W)\, H^{(1)}_{\eta} = 0. \tag{4}$$

Seeking a solution by the method of separation of variables, it is found that in the outer regions

$$h = \epsilon \sum_n \left({}_nA_{1\,n}\phi_1 + {}_nA_{2\,n}\phi_2\right) \sin \alpha_n \xi + \left({}_nB_{1\,n}\phi_1 + {}_nB_{2\,n}\phi_2\right) \cos \alpha_n \xi, \tag{5}$$

where ${}_n\phi_1$ and ${}_n\phi_2$ are solutions of

$${}_n\phi'' + 2(W'/W)\, {}_n\phi' - \alpha_n^2\, {}_n\phi = 0, \tag{6}$$

which is recognized as a variant of the Rayleigh equation. The constants A and B for region I cannot, at this stage, be related to those in the solution for region V, but the boundary condition that $h \to 0$ as $\eta \to \infty$ determines the ratios ${}_1A^{\mathrm{I}}_n/{}_2A^{\mathrm{I}}_n$ and ${}_1B^{\mathrm{I}}_n/{}_2B^{\mathrm{I}}_n$ while the boundary condition on the wavy surface determines the sums ${}_1A^{\mathrm{V}}_n + {}_1A^{\mathrm{V}}_n$ and ${}_1B^{\mathrm{V}}_n + {}_2B^{\mathrm{V}}_n$.

While the functions ϕ cannot, in general, be expressed in terms of known functions, it suffices for the purposes of this work to consider their asymptotic behaviour as $\eta \to 0$. If we let ${}_n\phi_1$ represent the solution which is regular around $\eta = 0$, it is possible to show by basic techniques that

$${}_n\phi_1 \sim 1 + O(\alpha_n \eta)^2 \quad \text{and} \quad {}_n\phi_2 \sim \frac{1}{\eta} - \ln \eta + O(\eta), \tag{7}$$

where, because interest is restricted to the real domain, the definition

$$\ln(-x) = \ln(x)$$

is invoked.

The order of magnitude expansion which led to (4) is valid only so long as $H^{(1)}$ and all of its derivatives are of $O(1)$ and from the asymptotic behaviour of ϕ_2 as $\eta \to 0$ it is clear that this restriction is violated near the critical layer. If it is argued that the large gradients of velocity associated with this behaviour will lead to important viscous stresses, then the techniques used to treat the Orr–Sommerfeld equation in the limit of infinite Reynolds number can be used to find a viscous solution which can be matched to the solutions which are valid in the outer regions. However, the point of view taken here is that if the viscosity is sufficiently small then viscous stresses may remain unimportant and the singularity in the outer solutions may be ascribed to the neglect of non-linear effects and may be overcome by matching the outer regions to inner regions in which certain non-linear terms are retained in the equation of motion.

In order to arrive at the appropriate equation of motion in the inner regions II and IV it is necessary to rescale both h and the length scale of the co-ordinate η according to

$$\begin{aligned} h &= \epsilon^{\frac{1}{2}}\hat{H}^{(1)}(\xi,\hat{\eta}) + O(\epsilon^{\frac{3}{4}}), \\ \hat{\eta} &= \epsilon^{-\frac{1}{2}}\eta, \\ \hat{W} &= \epsilon^{-\frac{1}{2}}W = \hat{\eta} - \epsilon^{\frac{1}{2}}\tfrac{1}{2}\hat{\eta}^2 + O(\epsilon). \end{aligned}$$

Substituting these forms into (2) and collecting terms of the same magnitude leads to the appropriate equation for the inner regions

$$\hat{W}\hat{H}^{(1)}_{\hat{\eta}\hat{\eta}} + \hat{W}_{\hat{\eta}}\{(1+\hat{H}^{(1)}_{\hat{\eta}})^3 - (1+\hat{H}^{(1)}_{\hat{\eta}})\} = 0.$$

Fortunately, this non-linear equation is essentially a first-order ordinary differential equation for $\hat{H}^{(1)}_{\hat{\eta}} + 1$ and is easily solved to yield

$$\hat{H}^{(1)}_{\hat{\eta}} + 1 = S\hat{W}\{\hat{W}^2 + f(\xi)\}^{-\frac{1}{2}}, \tag{8}$$

where the positive root is implied, S may be either ± 1 and $f(\xi)$ is arbitrary. Expressing $\hat{W}$ in terms of $\hat{\eta}$, this relation may be integrated and it is found that in the inner regions

$$h = \epsilon^{\frac{1}{2}}\{S\sqrt{[\hat{\eta}^2+f]} - \hat{\eta}\} - \epsilon\tfrac{1}{2}f\{\ln(\hat{\eta} + S\sqrt{[\hat{\eta}^2+f]}) - [S\hat{\eta}/\sqrt{(\hat{\eta}^2+f)}]\} + g(\xi), \tag{9}$$

where $g(\xi)$ is an arbitrary function which is, at most, $O(\epsilon^{\frac{1}{2}})$.

Turning now to the problem of matching the inner and outer solutions it becomes necessary to distinguish between the h_u, the solution valid above the closed streamline region, and h_l which pertains below. In each open streamline region the arbitrary functions f and g plus the sign S must be chosen such that, to $O(\epsilon)$, the outer expansion of the inner solution is identical to the inner expansion of the outer solution. From the asymptotic behaviour of $\phi(\eta)$ given in (7) it is seen that the inner representation of the outer solution is

$$h = \left\{\epsilon^{\frac{1}{2}}\frac{1}{\hat{\eta}} - \tfrac{1}{2}\epsilon\ln\epsilon - \epsilon\ln\hat{\eta}\right\}F(\xi) + \epsilon G(\xi),$$

where the functions

$$G(\xi) = \sum_n {}_nA_1 \sin\alpha_n\xi + {}_nB_1\cos\alpha_n\xi$$

and

$$F(\xi) = \sum_n {}_nA_2 \sin\alpha_n\xi + {}_nB_2\cos\alpha_n\xi$$

are, of course, different above and below the critical layer. Similarly, the outer representation of the inner solution given in (9) is

$$h = g(\xi) + S|\eta| - \eta + \epsilon\tfrac{1}{2}f\left\{\frac{S}{|\eta|} + \tfrac{1}{2}\ln\epsilon - \ln(\eta + S|\eta|) + S\frac{\eta}{|\eta|}\right\}.$$

Clearly a match can be achieved only if $S = \text{sign}(\eta)$, that is $+1$ for h_u and -1 for h_l. With this established, it is evident that the two expressions are identical if

$$f = 2F, \quad g(\xi) = -\epsilon \ln \epsilon \tfrac{1}{4} f + \epsilon\{\ln 2 - 1\} \tfrac{1}{2} f + \epsilon G. \tag{10}$$

Thus in each of the composite open streamline regions the functions f and g are determined once the corresponding functions F and G are known. Further, as mentioned earlier, the boundary condition at infinity places a constraint on F_u and G_u while the condition at the wavy surface provides a constraint on F_l and G_l. It remains, then, to relate the solution valid above the critical layer to that valid below the critical layer. In fact, is it only now that it becomes clear why there must be a special closed streamline region, for until the matching was considered it was not clear that a single 'inner' region would not adequately describe the neighbourhood of the critical layer. While the inability to achieve a satisfactory match with a single inner region is, in itself, sufficient evidence that there must be yet another region in which the inner scaling is inappropriate, the reason that the 'inner' scaling fails can be seen by noting that h_ξ becomes large as $\hat{\eta}^2 + f$ becomes small, as it must near the critical streamline $\hat{\eta} = 0$.

3. The closed streamline region

In the closed streamline region it is convenient to return to the Cartesian co-ordinates x and y in which the equation of motion (1) is

$$\psi_{yy} + \psi_{xx} = dW/d\eta.$$

In order to properly join this region to the open streamline regions on either side it is necessary to require that the bounding streamline, ψ_0, coincides with the streamlines which bound regions II and IV and, in addition, that the pressure be continuous across this boundary. In the absence of viscous stresses these two conditions imply that the velocity is continuous across ψ_0.

The appropriate choice for the vorticity distribution, specified by $W'(\eta)$, is not obvious and, in fact, if the flow is truly steady and inviscid there is no criterion by which a particular choice can be singled out. If the entire history of the flow were known the vorticity distribution would, of course, be determined by the initial distribution, but in order to determine the history of the flow it would be necessary to solve a very difficult non-linear, initial-value problem. If, however, the effects of a small but finite viscosity are taken into account it is found that there is an unambiguous choice for the function W'. This point was considered by Batchelor (1956) who showed that regardless of how small the viscosity becomes the viscous diffusion of vorticity will eventually lead to a distribution of vorticity which is uniform throughout the region except in a thin layer along the boundary. It should be emphasized that taking viscosity into account in this manner does not mean that the original objective of seeking a flow in which viscous stresses are negligible has been abandoned but rather that, in regions of closed streamline flow, the effects of viscosity cannot be ignored even when the viscous stresses are vanishingly small.

If the constant vorticity in region III is given the value Ω, (1) is seen to admit the solution

$$\psi = \tfrac{1}{2}\Omega\eta^2 = \tfrac{1}{2}\Omega\{y^2 + a_0 y + \Sigma\,({}_n a_1 \sinh\alpha_n y + {}_n a_2 \cosh\alpha_n y)\sin\alpha_n x + \Sigma\,({}_n b_1 \sinh\alpha_n y + {}_n b_2 \cosh\alpha_n y)\cos\alpha_n x\}.$$

Further, since it is expected that throughout this region $y = O(\epsilon^{\frac{1}{2}})$, the hyperbolic functions may be expanded in powers of y up to y^2 and the resulting quadratic equation solved for

$$y(x,\eta) = \eta + h = \frac{-\mathscr{G} \pm \{\mathscr{G}^2 + (1-\frac{1}{2}\mathscr{F}'')(\eta^2 - \mathscr{F})\}^{\frac{1}{2}}}{1-\frac{1}{2}\mathscr{F}''} + O(\epsilon^{\frac{3}{2}}), \tag{11}$$

where
$$\mathscr{G} = \tfrac{1}{2}\{a_0 + \Sigma\alpha_n({}_n a_1 \sin\alpha_n x + {}_n b_1 \cos\alpha_n x)\},$$
$$\mathscr{F} = \Sigma\,{}_n a_2 \sin\alpha_n x + {}_n b_2 \cos\alpha_n x.$$

The horizontal velocity is given by

$$u = \pm\,\Omega\{\mathscr{G}^2 + (1-\tfrac{1}{2}\mathscr{F}'')(\eta^2 - \mathscr{F})\}^{\frac{1}{2}}. \tag{12}$$

Having now obtained a solution describing the flow in the closed streamline region, all that remains is to join it to the solutions for the open streamline regions which were found in the previous section. This step involves requiring that u and h evaluated on the dividing streamline ψ_0 must be equal in the open and closed streamline regions. Equating u as determined from (8) to that given by (12) leads to

$$\hat{W}_0^u = -\hat{W}_0^l \equiv \hat{W}_0, \quad f^u = -f^l \equiv f, \quad \hat{W}_0^2 = |\text{minimum} f|$$

and
$$u_0 = \epsilon^{\frac{1}{2}}\{\hat{W}_0^2 + f\}^{\frac{1}{2}} = \Omega\{\mathscr{G}^2 + (1-\tfrac{1}{2}\mathscr{F}'')(\eta_0^2 - \mathscr{F})\}^{\frac{1}{2}} \equiv \epsilon^{\frac{1}{2}}\mathscr{H}(x), \tag{13}$$

where the subscript zero indicates evaluation on ψ_0 and the superscripts u and l refer to the upper and lower open streamline regions respectively. In equating h as given by (9) and (11) it is convenient to recall that $\hat{\eta} = \hat{W} + \epsilon^{\frac{1}{2}}\frac{1}{2}\hat{W}^2$ and to re-write (9) in terms of $\hat{W}$. Then noting from (13) that $\mathscr{F} = O(\epsilon)$, the continuity of h requires

$$h_0 = -\mathscr{G} - \eta_0 + S\epsilon^{\frac{1}{2}}(1/\Omega)\mathscr{H} = S\epsilon^{\frac{1}{2}}\mathscr{H} - \eta_0 + \epsilon\{\tfrac{1}{2}\hat{W}_0\mathscr{H} - \tfrac{1}{2}f\ln(\hat{W}_0 + \mathscr{H})\} + g, \tag{14}$$

where $S = 1$, $g = g^u$ along the upper portion of ψ_0 and $S = -1$, $g = g^l$ along the lower portion.

It is clear that, once Ω is determined, (13) and (14) relate f and g in the upper and lower 'inner' regions and these functions are, in turn, related to F and G of the 'outer' regions by (10). Finally, the boundary condition at the wavy surface and the limiting condition as $\eta \to \infty$ provide the additional restraints necessary to completely specify the solution. However, in order to determine Ω it is necessary to consider the viscous diffusion of vorticity into and out of the closed streamline region.

Across the bounding streamline ψ_0 the tangential velocity is continuous but according to the inviscid theory discussed above there is a jump in the vorticity of $O(\epsilon^{\frac{1}{2}})$. Along the upper half of ψ_0 the vorticity in the open streamline region is $1 - \epsilon^{\frac{1}{2}}\hat{W}_0$, along the lower half it is $1 + \epsilon^{\frac{1}{2}}\hat{W}_0$ and inside the closed streamline region it is Ω. This jump in vorticity will, in the limit of vanishing viscosity, be distributed across a thin boundary layer along ψ_0 and, unless Ω has adjusted to the proper value there will be a non-zero net transport into the closed streamline region thus precluding a truly steady flow.

In order to determine the value of Ω associated with a steady state it is necessary to examine the boundary layer along ψ_0. For this purpose it is convenient to introduce the co-ordinate l which measures distance along ψ_0 passing around the closed streamline region in the same direction as the flow. Introducing the usual boundary-layer approximations and making use of the Von Mises transformation the boundary layer may be described by

$$q\left(\frac{\partial \omega}{\partial l}\right)_{\psi} = \frac{1}{R} q \frac{\partial}{\partial \psi}\left[q \frac{\partial \omega}{\partial \psi}\right],$$

where ω is the vorticity, q is the dimensionless velocity directed along $\psi =$ constant and R is the Reynolds number which is large. While the vorticity varies significantly across the boundary layer, if the Reynolds number is large, the velocity q does not. The correct scaling in the boundary layer is $\hat{\psi} = \epsilon^{-\frac{1}{4}} R^{\frac{1}{2}} \psi$ and $q = \epsilon^{\frac{1}{2}} Q(l) + \epsilon^{\frac{1}{4}} R^{-\frac{1}{2}} \hat{Q}(l, \hat{\psi})$, where $\epsilon^{\frac{1}{2}} Q(l)$ is the 'free-stream' velocity predicted by the inviscid calculation. Substituting this into the boundary-layer equation and assuming $\epsilon^{\frac{1}{4}} R^{-\frac{1}{2}} \ll 1$ leads to

$$\left(\frac{\partial \omega}{\partial l}\right)_{\psi} = \frac{\partial^2}{\partial \hat{\psi}^2} Q\omega. \tag{15}$$

If we now consider integrating (15) along the two paths S_1 and S_2 depicted in figure 2, it is clear that, owing to the periodicity of the flow, the contribution from the vertical sections of S_2 will vanish and we may write

$$\frac{\partial^2}{\partial \hat{\psi}^2} \oint Q\omega\, ds = 0,$$

and, since $\partial \omega / \partial \hat{\psi}$ vanishes outside the boundary layer, this implies

$$\oint_{S_1} Q\omega\, ds = \oint_{S_2} Q\omega\, ds.$$

If S_1 is placed inside the inviscid core of the closed stream, the value of ω is Ω and similarly if S_2 is placed outside the boundary layer the vorticity along the upper portion is $1 - \epsilon^{\frac{1}{2}} \hat{W}_0$ and along the lower portion $\omega = 1 + \epsilon^{\frac{1}{2}} \hat{W}_0$. From (13) and (14) it is clear that, to $O(\epsilon)$, the integral $\int Q\, ds$ along the upper segment of S_2 is equal to that along the lower segment and, since the boundary layer is thin, is equal to one half the integral around S_1. Consequently

$$\Omega = \tfrac{1}{2}(1 - \epsilon^{\frac{1}{2}} \hat{W}_0) + \tfrac{1}{2}(1 + \epsilon^{\frac{1}{2}} \hat{W}_0) = 1,$$

that is the vorticity in the closed streamline region is the same as would exist on $\eta = 0$ in the absence of any wave-like disturbance.

With the value of Ω determined, it is now possible to simplify (13) and (14) to

$$\epsilon^{\frac{1}{2}} \{\hat{W}_0^2 + f\}^{\frac{1}{2}} = \{\mathscr{G}^2 + (\eta_0^2 - \mathscr{F})\}^{\frac{1}{2}}, \tag{16}$$

$$-\mathscr{G} = \epsilon\{\tfrac{1}{2} f \ln(\hat{W}_0 + \sqrt{[\hat{W}_0^2 + f]}) - \tfrac{1}{2}\hat{W}_0 \sqrt{[\hat{W}_0^2 + f]}\} - g, \tag{17}$$

where, in arriving at (16), notice has been taken of the fact that $\mathscr{F}'' = O(\epsilon)$. Equations (16) and (17) provide the final relationships required to complete the solution.

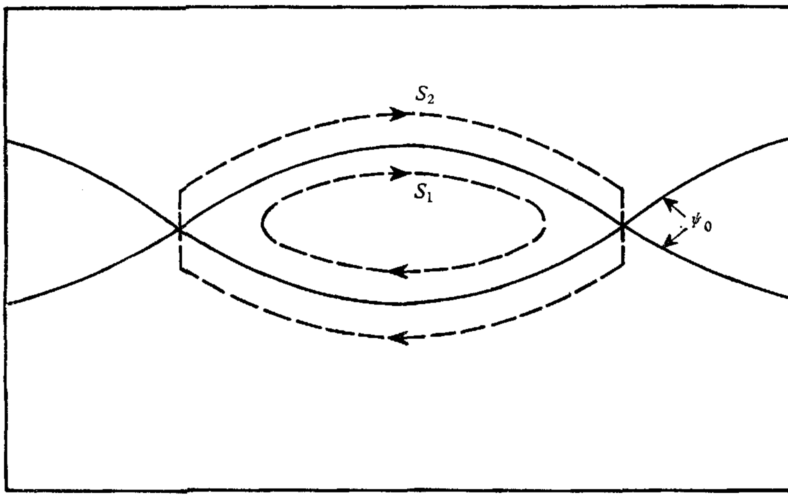

FIGURE 2. The integration paths used to find the vorticity in the closed streamline region.

The appropriate condition for relating the upper and lower 'outer' solutions is now easily seen. From (17) it follows that $g^u = g^l \equiv g$ while in (13) it has already been noted that $f^u = f^l \equiv f$. Returning to §2 in which the 'outer' and 'inner' solutions were matched, it follows from (10) that

$$F^u = F^l = \tfrac{1}{2}f \quad \text{and} \quad G^u = G^l = g/\epsilon + \tfrac{1}{4}f \ln \epsilon - (\ln 2 - 1)\tfrac{1}{2}f.$$

This result is to be contrasted with the equivalent matching condition obtained from the Orr–Sommerfeld equation which implies a jump in the phase of u across the critical layer which differs from 180°.

4. Summary

In the foregoing analysis an attempt has been made to investigate the nature of the critical layer associated with a wave-like disturbance in a high Reynolds number shear flow. It is well known that if interest is restricted to truly infinitesimal disturbances, the critical layer cannot be described without taking into account viscous stresses which, no matter how small the viscosity, are always important in that region. However, this analysis shows that if allowance is made for disturbances of finite amplitude, the critical layer need not be associated with non-vanishing viscous stresses and, in fact, as the Reynolds number of the shear flow approaches infinity the disturbance amplitude at which viscous stresses become negligible approaches zero.

The minimum amplitude for which this analysis can apply is determined primarily by the boundary layer which surrounds the closed streamline region.

In the previous section, the thickness of this boundary layer was seen to be $O(\epsilon^{-\frac{1}{4}}R^{-\frac{1}{2}})$ and for the analysis to be valid it is necessary that the boundary layer be thin compared with $O(\epsilon^{\frac{1}{2}})$ which is the dimension of the closed streamline region. Hence the inviscid model is valid only if

$$1 \gg R^{-\frac{1}{2}}\epsilon^{-\frac{3}{4}},$$

which is much more restrictive than the assumption made in solving the boundary-layer equations. In terms of dimensional variables this restriction is

$$a \gg (\nu^2 U_c'')^{\frac{1}{3}}/U_c',$$

where a is the wave amplitude and U_c' and U_c'' are the first and second derivatives of the mean velocity evaluated at the critical layer.

Admittedly a completely inviscid, steady-state model of shear flow over a wavy boundary must be regarded as of little more than academic interest owing to the fact that it cannot explain the generation of waves. To show this we need only consider the spatially averaged horizontal momentum equation for steady inviscid flow:

$$\frac{\partial}{\partial x}\overline{u^2}+\frac{\partial}{\partial y}\overline{uv}+\frac{\partial}{\partial x}\overline{p} = 0.$$

By hypothesis the flow is periodic in x and the velocity is everywhere continuous. As a result, the derivatives of $\overline{u^2}$ and $\overline{p}$ must vanish, the Reynolds stress is constant with height and, therefore, since far above the wave u and v vanish, the Reynolds stress is everywhere zero. Consequently, any theory which seeks to explain the generation of waves by a laminar shear flow must take into account viscous stresses.

While the complete neglect of viscous forces makes the inviscid model unsatisfactory as a description of real flows, the analysis does serve to establish two very important points. First, the inclusion of viscous stresses is not essential in arriving at a mathematically satisfactory description of wave-like perturbations to a shear flow. Second, the inclusion of non-linear terms in the 'perturbation' equations may lead to results which are significantly different from those obtained from a strictly linear model, particularly when the Reynolds number of the flow is large. These conclusions would appear to have important consequences in the theory of hydrodynamic stability which are deserving of further study.

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